Date 18/10/23

Saathi

Chapter-8

Binomial theorem

$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_n a^0 b^n$

$(a-b)^n = {}^nC_0 a^n b^0 - {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 - \dots + (-1)^n \, {}^nC_n b^n$

Ex = 8.1

Expand each of the following:-

2. $\left[\frac{2}{x} - \frac{x}{2}\right]^5$

$\Rightarrow {}^5C_0 \left[\frac{2}{x}\right]^5 - {}^5C_1 \left[\frac{2}{x}\right]^4 \left[\frac{x}{2}\right] + {}^5C_2 \left[\frac{2}{x}\right]^3 \left[\frac{x}{2}\right]^2 - {}^5C_3 \left[\frac{2}{x}\right]^2 \left[\frac{x}{2}\right]^3$
$+ {}^5C_4 \left[\frac{2}{x}\right] \left[\frac{x}{2}\right]^4 - {}^5C_5 \left[\frac{x}{2}\right]^5$

$\Rightarrow \frac{32}{x^5} - 5\left[\frac{16}{x^4}\right]\left[\frac{x}{2}\right] + 10\left[\frac{8}{x^3}\right]\left[\frac{x^2}{4}\right] - 10\left[\frac{4}{x^2}\right]\left[\frac{x^3}{8}\right] + 5\left[\frac{2}{x}\right]\left[\frac{x^4}{16}\right] - \frac{x^5}{32}$

$\Rightarrow \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5x^3}{8} - \frac{x^5}{32}$

1.) $(1-2x)^5$

$\Rightarrow {}^5C_0 (1)^5 - {}^5C_1 (1)^4 (2x) + {}^5C_2 (1)^3 - (2x)^2 - {}^5C_3 (1)^2 (2x)^3 + {}^5C_4 (1)^2 (2x)^4$
$- {}^5C_5 (2x)^5$

$\Rightarrow 1 - 5(2x) + 10(4x)^2 - 10(8x^3) + 5(16x^4) - (32x^5)$
$\Rightarrow 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

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3. $(2x-3)^6$

$\Rightarrow (2x-3)^6 = {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_1(2x)^4(3)^2 - {}^4C_3(2x)^3(3)^3$

$\Rightarrow 64x^6 - 6(32x^5)(3) + 15(16x^4)(9) - 20(8x^3)(27) + 15(4x^2)(81) - 6(2x)(243) + 729$

$\Rightarrow 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$

4. $\left[\frac{x}{3} + \frac{1}{x}\right]^5$

$\Rightarrow {}^5C_0\left[\frac{x}{3}\right]^5 + {}^3C_1\left[\frac{x}{3}\right]^4\left[\frac{1}{x}\right] + {}^3C_2\left[\frac{x}{3}\right]^3\left[\frac{1}{x}\right]^2$

$\Rightarrow \frac{x^5}{243} + 5\left[\frac{x^4}{81}\right]\left[\frac{1}{x}\right] + 10\left[\frac{x^3}{27}\right]\left[\frac{1}{x^2}\right] + 10\left[\frac{x^2}{9}\right]\left[\frac{1}{x^3}\right] + 5\left[\frac{x}{3}\right]\left[\frac{1}{x^4}\right] + \left[\frac{1}{x^5}\right]$

$\frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^3}$

5. $\left[x + \frac{1}{x}\right]^6$

$\Rightarrow {}^6C_0(x)^6 + {}^6C_1(x)^1\left[\frac{1}{x}\right] + {}^6C_2(x)^4\left[\frac{1}{x}\right]^2 + {}^6C_3(x)^3\left[\frac{1}{x}\right]^3 + {}^6C_4(x)^2\left[\frac{1}{x}\right]^4 + {}^6C_3(x)\left[\frac{1}{x}\right]^5 + {}^6C_6\left[\frac{1}{x}\right]^6$

$\Rightarrow x^4 + 6(x)^3\left[\frac{1}{x}\right] + 15(x)^4\left[\frac{1}{x^2}\right] + 20(x)^3\left[\frac{1}{x^3}\right] + 15(x^2)\left[\frac{1}{x^4}\right] + 6(x)\left[\frac{1}{x^5}\right] + \frac{1}{x^6}$

$\Rightarrow x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

6. $(96)^3 = (100-4)^3$

$\Rightarrow {}^3C_0(100)^3 - {}^3C_1(100)^2(4) - {}^3C_2(100)(4)^2 - {}^3C_3(4)^3$

$\Rightarrow (100)^3 - 3(100)^2(4) + 3(100)(4)^2 - (4)^3$

$\Rightarrow 1000000 - 120000 + 4800 - 64 = 884736$ (Ans)

7. $(102)^5 = (100+2)^5$

$\Rightarrow {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3 + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5$

$\Rightarrow (100)^5 + 5(100)^4(2) + 10(100)^3(2)^2 + 5(100)(2)^3 + 5(100)(2)^4 + (2)^5$

$\Rightarrow 10000000000 + 1000000000 + 40000000 + 80000 + 8000 + 32$

$\Rightarrow 11040808032$ (Ans)

8. $(101)^4 = (100+1)^4$

$\Rightarrow {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(1)^4$

$\Rightarrow (100)^4 + 4(100)^3 + 6(100)^2 + 4(100) + (1)^4$

$\Rightarrow 100000000 + 4000000 + 60000 + 400 + 1$

$\Rightarrow 104060401$ (Ans)

9. $(99)^5 = (100-1)^5$

$\Rightarrow {}^5C_0(100)^5 - {}^5C_1(100)^4(1) + {}^5C_2(100)^3(1)^2 - {}^5C_3(100)^2(1)^3 + {}^5C_4(100)(1)^4 - {}^5C_5(1)^5$

$\Rightarrow (100)^5 - 5(100)^4 + 10(100)^3 - 10(100)^2 + 5(100) - 1$

$\Rightarrow 10000000000 - 5(100000000) + 10(1000000) - 10(10000) + 500 - 1$

$\Rightarrow 9509900499$ (Ans)

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10. Using Binomial theorem, indicate which no. is larger $(1.1)^{10000}$ or 1000

$(1.1)^{10000} = (1+0.1)^{10000}$

$\Rightarrow (1+0.1)^{10000} \; {}^{10000}C_1 (1.1)$

$\Rightarrow 1+10000 \times 1.1$

$\Rightarrow 1+11000$

$\Rightarrow 11001$

$\Rightarrow (1.1)^{10000} > 1000$

11. Find $(a+b)^4 - (a-b)^4$. Hence evaluate $(\sqrt{3}+\sqrt{2})^4 - (\sqrt{3}-\sqrt{2})^4$

$(a+b)^4 = {}^4C_0 a^4 + {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 + {}^4C_3 ab^3 + {}^4C_4 b^4$

$(a-b)^4 = {}^4C_0 a^4 - {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 - {}^4C_3 ab^3 + {}^4C_4 b^4$

$\Rightarrow (a+b)^4 - (a-b)^4 = {}^4C_0 a^4 + {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 + {}^4C_3 ab^3 +$
${}^4C_4 b^4 - [{}^4C_0 a^4 - {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 - {}^4C_3 ab^3 + {}^4C_4 b^4]$

$\Rightarrow 2({}^4C_1 a^3 b + {}^4C_3 ab^3)$

$\Rightarrow 2(4a^3 b + 4ab^3)$

$\Rightarrow 8ab(a^2+b^2)$

Substituting $a=\sqrt{3}$ $b=\sqrt{2}$

$(\sqrt{3}+\sqrt{2})^4 - (\sqrt{3}-\sqrt{2})^4 = 8(\sqrt{3})(\sqrt{2})\{(\sqrt{3})^2 + (\sqrt{2})^2\}$

$\Rightarrow 8(\sqrt{6})(3+2)$

$\Rightarrow 40\sqrt{6}$ (Ans)

12. Find $(x+1)^6 + (x-1)^6$. Hence, or otherwise evaluate $(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$

$\Rightarrow (x+1)^6 = {}^6C_0 x^6 + {}^6C_1 x^5 + {}^6C_2 x^4 + {}^6C_3 x^3 + {}^6C_4 x^2 + {}^6C_5 x + {}^6C_6$

$\Rightarrow (x-1)^6 = {}^6C_0 x^6 - {}^6C_1 x^5 + {}^6C_2 x^4 + {}^6C_3 x^3 + {}^6C_4 x^2 - {}^6C_5 x + {}^6C_6$

$\Rightarrow (x+1)^6 - (x-1)^6 = {}^6C_0 x^6 + {}^6C_1 x^5 + {}^6C_2 x^4 + {}^6C_3 x^3 + {}^6C_4 x^2 + {}^6C_5 x +$
${}^6C_6 - [{}^6C_0 x^6 - {}^6C_1 x^5 + {}^6C_2 x^4 - {}^6C_3 x^3 + {}^6C_4 x^2 - {}^6C_5 x + {}^6C_6]$

$\Rightarrow 2[{}^6C_0 x^6 + {}^6C_2 x^4 + {}^6C_4 x^2 + {}^6C_6]$

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$\Rightarrow 2[x^6+15x^4+15x^2+1]$

Substituting $x=\sqrt{2}$

$(\sqrt{2}+1)^6-(\sqrt{2}-1)^6 = 2[(\sqrt{2})^6+15(\sqrt{2})^4+15(\sqrt{2})^2+1]$

$\Rightarrow 2[8+15\times4+15\times2+1]$

$\Rightarrow 2(8+60+30+1) \Rightarrow 2\times99 = 198$ (Ans)

13. Show that $9^{n+1}-8n-9$ is divisible by 64 whenever n is a +ve integer.

$\Rightarrow (1+a)^m = {}^mC_0 + {}^mC_1 a + {}^mC_2 a^2 + \dots + {}^mC_m a^m$

$\Rightarrow a=8 \quad m=n+1$

$\Rightarrow (1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1 8 + {}^{n+1}C_2 (8)^2 + \dots + {}^{n+1}C_{n+1}(8)^{n+1}$

$\Rightarrow 9^{n+1} = 1+(n+1)8+8^2[{}^{n+1}C_2 + {}^{n+1}C_3(8) + \dots + {}^{n+1}C_{n+1}(8)^{n-1}]$

$\Rightarrow 9^{n+1} = 9+8n+64[{}^{n+1}C_2 + {}^{n+1}C_3(8) + \dots + {}^{n+1}C_{n+1}(8)^{n-1}]$

$\Rightarrow 9^{n-1}-8n-9 = 64K$ proved

14. Prove that

$$\sum_{r=0}^{n} 3^r \, {}^nC_r = 4^n$$

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} a^{n-r} b^r = (a+b)^n$$

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} (1)^{n-r} (3)^r = (1+3)^n$$

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} (1)(3)^r = 4^n$$

proved

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Miscellaneous Exercise

Q4- If - - - - - - - - - - - - - - integer

To prove :- $(a-b)$ is a factor of $a^n - b^n$ we need to prove
$a^n - b^n = k(a-b)$

Here $a = a-b+b$

$a^n = (a-b+b)^n$

$\Rightarrow a^n = {}^nC_0 (a-b)^n b^0 + {}^nC_2 (a-b)^{n-1} b^1 + \dots + {}^nC_{n-1} (a-b) b^{n-1} + {}^nC_n (a-b)^0 b^n$

$\Rightarrow a^n = (a-b)^n + {}^nC_2 (a-b)^{n-1} + b + \dots + {}^nC_{n-1} (a-b) b^{n-1} + b^n$

$\Rightarrow a^n = (a-b)\{(a-b)^{n-1} + {}^nC_2 (a-b)^{n-2} b + \dots + {}^nC_{n-1} b^{n-1}\} + b^n$

$\Rightarrow a^n - b^n = (a-b)[(a-b)^{n-1} + {}^nC_2 (a-b)^{n-2} b + \dots + {}^nC_{n-1} b^{n-1}]$

$\Rightarrow a^n - b^n = (a-b) k$

where $k = (a-b)^{n-1} + {}^nC_2 (a-b)^{n-2} b + \dots {}^nC_{n-1} b^{n-1}$

Hence proved $(a-b)$ is a factor of $a^n - b^n$

Q5- Evaluate - - - - - - - - - - - $(\sqrt{3}-\sqrt{2})^6$

$\Rightarrow {}^6C_0(\sqrt{3})^6(\sqrt{2})^0 + {}^6C_1(\sqrt{3})^5(\sqrt{2})^1 {}^6C_2(\sqrt{3})^4(\sqrt{2})^2 + {}^6C_3(\sqrt{3})^3(\sqrt{2})^3 + {}^6C_4 (\sqrt{3})^2(\sqrt{2})^4 + {}^6C_5(\sqrt{3})^1(\sqrt{2})^5 + {}^6C_6(\sqrt{3})^0(\sqrt{2})^6 - [{}^6C_0(\sqrt{3})^6(-\sqrt{2})^0 + {}^6C_1(\sqrt{3})^5(-\sqrt{2})^1 + {}^6C_2(\sqrt{3})^4(-\sqrt{2})^2 + {}^6C_3(\sqrt{3})^3(-\sqrt{2})^3 + {}^6C_4(\sqrt{3})^2(-\sqrt{2})^4 + {}^6C_5(\sqrt{3})^1(-\sqrt{2})^5 + {}^6C_6(\sqrt{3})^0(-\sqrt{2})^6]$

$\Rightarrow 1 \times 27 \times 1 + 6 \times 9\sqrt{3} \times \sqrt{2} + 15 \times 9 \times 2 + 20 \times 3\sqrt{3} \times 2\sqrt{2} + 15 \times 3 \times 4 + 6 \times \sqrt{3} \times 4\sqrt{2} + 1 \times 1 \times 8 - [1 \times 27 \times 1 - 6 \times 9\sqrt{3} \times \sqrt{2} + 15 \times 9 \times 2 - 20 \times 3\sqrt{3} \times 2\sqrt{2} + 15 \times 3 \times 4 - 6 \times \sqrt{3} \times 4\sqrt{2} + 1 \times 1 \times 8]$

$\Rightarrow \cancel{27} + 54\sqrt{6} + \cancel{270} + 120\sqrt{6} + \cancel{180} + 24\sqrt{6} + \cancel{8} - \cancel{27} + 54\sqrt{6} - \cancel{270} + 120\sqrt{6} - \cancel{180} + 24\sqrt{6} - \cancel{8}$

$\Rightarrow 54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6} + 54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}$

$\Rightarrow 198\sqrt{6} + 198\sqrt{6} \quad \Rightarrow \boxed{(\sqrt{3}+\sqrt{2})^6 - (\sqrt{3}-\sqrt{2})^6 = 396\sqrt{6}}$ (Ans)

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Q6- Find - - - - - - - - - - - - - - $(a^2-\sqrt{a^2-1})^4$

$$\Rightarrow [{}^4C_0x^4+{}^4C_1x^3y+{}^4C_2x^2y^2+{}^4C_3xy^3+{}^4C_4y^4]+[{}^4C_0x^4-{}^4C_1x^3y+{}^4C_2x^2y^2-{}^4C_3xy^3+{}^4C_4y^4]$$

$$\Rightarrow 2[{}^4C_0x^4+{}^4C_2x^2y^2+{}^4C_4y^4]$$

$$\Rightarrow 2(x^4+6x^2y^2+y^4)$$

$$(x+y)^4+(x-y)^4 = 2(x^4+6x^2y^2+y^4) - (1)$$

Putting $x=a^2$ and $y=\sqrt{a^2-1}$ in eq (1)

$$(a^2+\sqrt{a^2-1})^4+(a^2-\sqrt{a^2-1})^4$$

$$\Rightarrow 2[((a^2)^4+6(a^2)^2(\sqrt{a^2-1})^2+(\sqrt{a^2-1})^4]$$

$$\Rightarrow 2[a^8+6a^4(a^2-1)+(a^2-1)^2]$$

$$\Rightarrow 2[a^8+6a^6-6a^4+a^4-2a^2+1]$$

$$\Rightarrow 2a^8+12a^6-10a^4-4a^2+2$$

Q8- Find - - - - - - - - - - - - - - - expansion of

$$\left[\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right]^n = \left[2^{1/4}+\frac{1}{3^{1/4}}\right]^n$$

put $2^{1/4}=x$ and $\frac{1}{3^{1/4}}=y$

$$\frac{T_5 \text{ of } (x+y)^n}{T_5 \text{ of } (y+x)^n} = \sqrt{6}$$

$$\frac{{}^nC_4\,x^{n-4}y^4}{{}^nC_4\,y^{n-4}x^4} = \sqrt{6}$$

$$\frac{x^{n-4-4}}{y^{n-4-4}} = \sqrt{6} \Rightarrow \frac{x^{n-8}}{y^{n-8}} = \sqrt{6}$$

$$\left[\frac{x}{y}\right]^{n-8} = \sqrt{6}$$

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Putting $x = 2^{1/4}$ and $y = \frac{1}{3^{1/4}}$, we have

$$(2^{1/4} \cdot 3^{1/4})^{n-8} = \sqrt{6}$$
$$(6^{1/4})^{n-8} = 6^{1/2}$$
$$6^{\frac{n-8}{4}} = 6^{1/2} \Rightarrow \frac{n-8}{4} = \frac{1}{2}$$

$$2n - 16 = 4$$
$$2n = 4 + 16$$
$$n = 10$$

Q7- Find - - - - - - - - - - - - - - expansion

$$(0.99)^5 = (1 - 0.01)^5$$
$$= {}^5C_0 - {}^5C_1(0.01) + {}^5C_2(0.01)^2$$
$$= 1 - 5(0.01) + 10(0.0001)$$
$$= 1 - 0.05 + 0.001$$
$$= 1.001 - 0.050$$
$$= 0.951 \text{ (Ans)}$$

Q9- Expand - - - - - - - - - - - - - $x \neq 0$

$$\left[1 + \frac{x}{2} - \frac{2}{x}\right] = \left[1 + \left[\frac{x}{2} - \frac{2}{x}\right]\right]^4$$

$$\Rightarrow {}^4C_0 + {}^4C_1\left[\frac{x}{2} - \frac{2}{x}\right] + {}^4C_2\left[\frac{x}{2} - \frac{2}{x}\right]^2 + {}^4C_3\left[\frac{x}{2} - \frac{2}{x}\right]^3 + {}^4C_4\left[\frac{x}{2} - \frac{2}{x}\right]^4$$

$$\Rightarrow \left[\because (1+y)^n = {}^nC_0 + {}^nC_1 y + {}^nC_2 y^2 + {}^nC_3 y^3 \ldots + {}^nC_n y^n \quad y = \frac{x}{2} - \frac{2}{x}\right]$$

$$\Rightarrow 1 + 4\left[\frac{x}{2} - \frac{2}{x}\right] + 6\left[\left[\frac{x}{2}\right]^2 - 2\left[\frac{x}{2}\right]\left[\frac{2}{x}\right] + \left[\frac{2}{x}\right]^2\right] + 4\left[{}^3C_0\left[\frac{x}{2}\right]^3 - {}^3C_1\left[\frac{x}{2}\right]^2\left[\frac{2}{x}\right] + {}^3C_2\left[\frac{x}{2}\right]\left[\frac{2}{x}\right]^2 - {}^3C_3\left[\frac{2}{x}\right]^3\right] + \left[{}^4C_0\left[\frac{x}{2}\right]^4 - {}^4C_1\left[\frac{x}{2}\right]^3\left[\frac{2}{x}\right] + {}^4C_2\left[\frac{x}{2}\right]^2\left[\frac{2}{x}\right]^2 - {}^4C_3\left[\frac{x}{2}\right]\left[\frac{2}{x}\right]^3 + {}^4C_4\left[\frac{2}{x}\right]^4\right]$$

$$\Rightarrow 1+2x-\frac{8}{x}+6\left[\frac{x^2}{4}-2+\frac{4}{x^2}\right]+4\left[\frac{x^3}{8}-3\left[\frac{x^2}{4}\right]\left[\frac{2}{x}\right]+3\left[\frac{x}{2}\right]\left[\frac{4}{x^2}\right]-\frac{8}{x^3}\right]+\left[\frac{x^4}{16}-4\left[\frac{x^3}{8}\right]\left[\frac{2}{x}\right]+6\left[\frac{x^2}{4}\right]\left[\frac{4}{x^2}\right]-4\left[\frac{x}{2}\right]\left[\frac{8}{x^3}\right]+\frac{16}{x^4}\right]$$

$$\Rightarrow 1+2x-\frac{8}{x}+\frac{3x^2}{2}-12+\frac{24}{x^2}+\frac{x^3}{2}-6x+\frac{24}{x}-\frac{32}{x^3}+\frac{x^4}{16}-x^2+6-\frac{16}{x^2}+\frac{16}{x^4}$$

$$\Rightarrow \frac{16}{x^4}-\frac{32}{x^3}+\frac{8}{x^2}+\frac{16}{x}-5-4x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}$$

10. Find - - - - - - - - - - - - - - theorem.

$$\Rightarrow (3x^2-2ax+3a^2)^3=[(3x^2-2ax)+3a^2]^3$$

$$={}^3C_0(3x^2-2ax)^3+{}^3C_1(3x^2-2ax)^2(3a^2)^1+{}^3C_2(3x^2-2ax)^1(3a^2)^2+{}^3C_3(3a^2)^3$$

$$\Rightarrow [{}^3C_0(3x^2)^3-{}^3C_1(3x^2)^2(2ax)+{}^3C_2(3x^2)(2ax)^2-{}^3C_3(2ax)^3]+3(9x^4-12ax^3+4a^2x^2)(3a^2)+3(3x^2-2ax)(9a^4)+27a^6$$

$$\Rightarrow 27x^6-54ax^5+36a^2x^4-8a^3x^3+81a^2x^4-108a^3x^3+36a^4x^2+81a^4x^2-54a^5x+27a^6$$

$$\Rightarrow 27x^6-54ax^5+117a^2x^4-116a^3x^3+117a^4x^2-54a^5x+27a^6$$